Ch 10 Video Notes

Thursday, January 4, 2024 8:30 AM

list of #s with a pattern

10.1 Notes: Sequences and Summation Notation

Fibonacci Sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

$a_1 = 1$ $a_4 = 1+2 = 3$

$a_2 = 1$ $a_5 = 2+3 = 5$

$a_3 = 1+1 = 2$

An infinite sequence $\{a_n\}$ is a function whose domain is the set of positive integers. The function values, or terms, of the sequence are represented by $a_n \to$ general term

without end $a_1 = 1^{st}$ term

Sequences whose domains consist only of the first n positive integers are called finite sequence.

(ends)

Example 1: Write the first 4 terms of each sequence.

a) $a_n = 2n + 5$ b) $a_n = \frac{(-1)^n}{2^n+1}$

general term

$a_1 = 2(1) + 5 = 7$

$a_2 = 2(2) + 5 = 9$

$a_3 = 2(3) + 5 = 11$

$a_4 = 2(4) + 5 = 13$

$\boxed{7, 9, 11, 13}$

$a_1 = \frac{(-1)^1}{2^1+1} = \frac{-1}{3}$ $a_3 = \frac{(-1)^3}{2^3+1} = \frac{-1}{9}$

$a_2 = \frac{(-1)^2}{2^2+1} = \frac{1}{5}$ $a_4 = \frac{(-1)^4}{2^4+1} = \frac{1}{17}$

$\boxed{-\frac{1}{3}, \frac{1}{5}, -\frac{1}{9}, \frac{1}{17}}$

Recursion Formula:

* starting value is given ; * next terms are relatived to previous term

$a_1 = 5$; $\boxed{a_n = 2a_{n-1} + 1}$ $n \geq 2$

previous term

Example 2) Find the first 4 terms of the sequence in which $a_1 = 3$, $a_n = 2a_{n-1} + 5$, for $n \geq 2$.

$a_1 = 3$

$a_2 = 2 \cdot 3 + 5 = 11$

$a_3 = 2 \cdot 11 + 5 = 27$

$a_4 = 2 \cdot 27 + 5 = 59$

$\boxed{3, 11, 27, 59}$

Factorial Notation:

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$6! = 6 (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 720$

$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$

$(n+1)! = (n+1) \cdot n \cdot (n-1) \cdot \ldots \cdot 1$

Example 3: Write the first 4 terms of $a_n = \frac{20}{(n+1)!}$.

$a_1 = \frac{20}{(1+1)!} = \frac{20}{2 \cdot 1} = 10$

$a_2 = \frac{20}{(2+1)!} = \frac{\cancel{20}\,10}{3 \cdot \cancel{2} \cdot 1} = \frac{10}{3}$

$a_3 = \frac{20}{(3+1)!} = \frac{20}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{20}{24} = \boxed{\frac{5}{6}}$

$a_4 = \frac{20}{6!} = \frac{\cancel{20}\,\cancel{4} \cdot 1}{\cancel{5} \cdot \cancel{4} \cdot 3 \cdot 2 \cdot 1} = \boxed{\frac{1}{6}}$

$\boxed{10, \frac{10}{3}, \frac{5}{6}, \frac{1}{6}}$

Example 4: Evaluate each factorial expression:

a) $\frac{14!}{2!12!} = \frac{7 \cdot \cancel{14} \cdot 13 \cdot \cancel{12!}}{\cancel{2} \cdot 1 \cdot \cancel{12!}} = \boxed{91}$

b) $\frac{n!}{(n-1)!} = \frac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = \boxed{n}$

c) $\frac{(n+2)!}{n+2} = \frac{\cancel{(n+2)}(n+1)!}{\cancel{(n+2)}} = \boxed{(n+1)!}$

Summation Notation:

Sum → Addition

$\sum_{i=1}^{n} a_i \longrightarrow a_1 + a_2 + a_3 + \dots a_n$

Sigma (Greek letter) $\boxed{\text{SUM ADD}}$

n ← ending value to plug in for last term

a_i general term

$i=1$ ← starting value to "plug in" for the first term

Series
summation $\sum$ } adding up the terms of a sequences

Example 5: Expand and evaluate each series. → add up the terms of a sequence (summation)

a) $\sum_{i=1}^{6} 2i^2 = 2(1)^2 + 2(2)^2 + 2(3)^2 + 2(4)^2 + 2(5)^2 + 2(6)^2$

$= (2 + 8) + (18 + 32) + (50) + (72)$

$= 10 + 50 + 50 + 72 = 182$

b) $\sum_{k=3}^{5} (2^k - 3) = \underbrace{2^3 - 3} + \underbrace{2^4 - 3} + \underbrace{2^5 - 3}$

$= 5 + 13 + 29$

$= \boxed{47}$

c) $\sum_{i=1}^{5} 4 = 4 + 4 + 4 + 4 + 4$

$= \boxed{20}$

Example 6: Express each sum using summation notation:

a) $1^2 + 2^2 + 3^2 + \dots + 9^2$

$\sum_{n=1}^{9} n^2$

b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}}$ ← general term

$\sum_{i=1}^{n} \frac{1}{2^{i-1}}$ (infinite series)

c) $25 + 50 + 75 + \dots + 250$

$25 \cdot 1 + 25 \cdot 2 + 25 \cdot 3 + \dots + 25 \cdot 10$

$\boxed{\sum_{n=1}^{10} 25n}$

d) $a + (a+d) + (a+2d) + \dots + (a+nd)$

$(a + 0 \cdot d) + (a + 1 \cdot d) + (a + 2 \cdot d) + \dots + (a + n \cdot d)$

$\boxed{\sum_{k=0}^{n} a + kd}$ or $\boxed{\sum_{k=1}^{n+1} a + (k-1)d}$

*

10.2 Notes: Arithmetic Sequences

Sample: $\underbrace{8, 11, 14, 17, 20, 23, \ldots}_{d=3}$

Arithmetic sequence:
pattern +/− same time

sample: −5, −7, −9, −11, −13, …
d = −2 −2 −2 −2

What is a **common difference**? d

amount added to each term to create the sequence

Examples: Write the first six terms of each arithmetic sequence.

1) $a_1 = 6$ and $a_n = a_{n-1} - 2$

$a_1 = 6$
$a_2 = 6 - 2 = 4$
$a_3 = 4 - 2 = 2$
$a_4 = 2 - 2 = 0$
$a_5 = -2$
$a_6 = -4$

term	0	1	2	3	4	5	6
	8	6	4	2	0	−2	−4

2) $a_1 = 100$ and $a_n = a_{n-1} + 30$

$a_1 = 100$
$a_2 = 100 + 30 = 130$
$a_3 = 130 + 30 = 160$
$a_4 = 190$
$a_5 = 220$
$a_6 = 250$

$\boxed{100, 130, 160, 190, 220, 250}$

General term of an Arithmetic Sequence (also called "explicit form")

$\boxed{a_n = dn + a_0}$ OR $a_n = a_1 + d(n-1)$

general term; common diff; input variable; # prior to the sequence begin

Examples:

3) Find the ninth term of the arithmetic sequence with a first term of 6 and a common difference of -5.

$a_1 = 6$
$d = -5$

$a_n = dn + a_0$
$\boxed{a_n = -5n + 11}$
$a_9 = -5(9) + 11 = -45 + 11$
$\boxed{a_9 = -34}$

11 { 6, 1, −4, −9, …
a_1 a_2 a_3

$a_n = a_1 + d(n-1)$
$a_n = 6 + -5(n-1)$
$a_n = 6 - 5n + 5 \rightarrow \boxed{a_n = -5n + 11}$

4) Find the eighth term of the arithmetic sequence with a first term of 4 and a common difference of 7.

$a_1 = 4$
$d = 7$

$a_n = dn + a_0$
$a_n = 7n - 3$
$a_8 = 7(8) - 3$
$\boxed{a_8 = 53}$

−3 { 4, 11, 18, 25, 32, 39, 46, 53
a_0 a_1 a_2 8th

Example 5) Teachers in the US earned an average of \$44,600 in 2002. This amount has increased by approximately \$1130 per year.

a_0

$a_1 \to 2003$ yr

a) Write a formula for the nth term of the arithmetic sequence that describes teachers' average earnings n years after 2002.

$a_n = dn + a_0$

$a_n = 1130n + 44,6000$

b) How much will US teachers earn, on average, by the year 2025?

$a_{23} = 1130(23) + 44,600$

$\boxed{= \$70,590}$

?, 32, 32.7, 33.4,

Example 6) Americans are eating more meals behind the wheel. In 2004, we averaged 32 a la car meals per year, which is increasing by approximately 0.7 meal per year.

$a_1\ (2004) = 32$

$d = 0.7$

a) Write a formula for the nth term of the arithmetic sequence that models the average number of car meals n years after 2003.

$a_n = dn + a_0$

$a_n = 0.7n + 31.3$

$a_n = a_1 + d(n-1)$

$a_n = 32 + 0.7(n-1)$

$a_n = 32 + 0.7n - 0.7$

$a_n = 0.7n + 31.3$

b) How many car meals will Americans average by the year 2023?

$2023 - 2003 = 20$

$a_{20} = 0.7(20) + 31.3$

$= \boxed{45.3 \text{ meals}}$

Sum of the First n Terms of an Arithmetic Sequence:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

a_1 is the first term

a_n is the nth term

partial sum "series"

Σ

Example 7) Find the sum of the first 100 terms of the arithmetic sequence: 1, 3, 5, 7, …

-1

$S_{100} = \frac{100}{2}(1 + 199)$

$= 50(200) = \boxed{10,000}$

$a_n = 2d -$

$a_{100} = 2(100) - 1$

$= 199$

Example 8) Find the sum of the first 15 terms of the arithmetic sequence: 3, 6, 9, 12, ….

$S_{15} = \frac{15}{2}(3 + 45)$

$= \boxed{360}$

$a_n = 3n + 0$

$a_{15} = 3 \cdot 15$

$= 45$

Example 9) Find the following sum: $\sum_{i=1}^{25}(5i-9)$

general form

$-4, 1, 6, \ldots$

$a_{25} = 5(25) - 9$
$= 125 - 9$
$= 116$

$S_{25} = \frac{25}{2}(-4 + 116)$
$= \boxed{1400}$

Example 10) Find the following sum: $\sum_{i=1}^{30}(6i-11)$

$a_1 = 6(1) - 11 = -5$
$a_{30} = 6(30) - 11 = 169$

$S_{30} = \frac{30}{2}(-5 + 169)$
$= \boxed{2460}$

Example 11) Your grandmother has assets of \$500,000. One option that she is considering involves an adult residential community for a six-year period beginning in 2009. The model $a_n = 1800n + 64130$ describes yearly adult residential community costs n years after 2008.

$a_1 (2009) = 65,930$
$d = 1800$

a) Does your grandmother have enough to pay for the facility for six years?

$S_6 = \frac{6}{2}(65,930 + 74,930)$

$a_6 (2014) = 1800(6) + 64130$
$= 74930$

$\boxed{S_6 = \$422,580}$ Yes, there is enough to pay for 6yrs.

b) How much would it cost for the adult residential community for a ten-year period beginning in 2009?

$S_{10} = \frac{10}{2}(65,930 + 82,130)$
$= \boxed{\$740,300}$

$a_{10} = 1800(10) + 64130$

10.3 Notes: Geometric Sequences

A sequence in which each term after the first is obtained by multiplying the preceding term by a fixed nonzero constant. The amount by which we multiply each time is called the common ratio (r) of the sequence.

$3, 6, 12, 24, 48, \ldots$ $\quad r = \frac{a_2}{a_1} = \frac{6}{3} = 2$

$48, -16, \frac{16}{3}, -\frac{16}{9}, \frac{16}{27}, \ldots$ $\quad r = \frac{-16}{48} = -\frac{1}{3}$

$r = -\frac{1}{3}$ $\quad r = -\frac{1}{3}$

Example 1) Write the first six terms of the geometric sequence with first term of 12 and a common ratio of $\frac{1}{2}$.

$12, 6, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}$

$\cdot\frac{1}{2}$ $\cdot\frac{1}{2}$ $\cdot\frac{1}{2}$ $\cdot\frac{1}{2}$

General term of a Geometric Sequence: $a_n = a_1(r)^{n-1}$ note: $(r \neq 1)$

general term $\quad$ 1st term $\quad$ common ratio

Examples:

2) Find the eighth term of the geometric sequence whose first term is -4 and whose common ratio is -2.

$a_n = -4(-2)^{n-1}$ $\quad a_1 = -4$

$a_8 = -4(-2)^7$ $\quad r = -2$

$\boxed{a_8 = 512}$

3) Find the seventh term of the geometric sequence whose first term is 5 and whose common ratio is $\frac{1}{3}$.

$a_n = 5\left(\frac{1}{3}\right)^{n-1}$ $\quad a_1 = 5$

$a_7 = 5\left(\frac{1}{3}\right)^6 = \frac{5}{1}\cdot\frac{1}{729} = \boxed{\frac{5}{729}}$ $\quad r = \frac{1}{3}$

or $\boxed{\approx 0.007}$

4) Write the general term for the geometric sequence 3, 6, 12, 24, 48, … Then use the formula for the general term to find the 12^{th} term.

$a_1 = 3$ $\quad r = 2$

$\boxed{a_n = 3(2)^{n-1}}$

$a_{12} = 3(2)^{11}$

$= \boxed{6144}$

Sum of the First n Terms of a Geometric Sequence: $S_n = \frac{a_1(1-r^n)}{1-r}$ note: $(r \neq 1)$

1st term

common

partial sum

common ratio

For Examples 5 – 8, find the requested sum.

5) Find the sum of the first 18 terms of the geometric sequence: $2, -8, 32, -128, \ldots$

$r = \frac{a_2}{a_1} = \frac{-8}{2}$

$r = -4$

$S_{18} = \frac{2(1-(-4)^{18})}{1-(-4)} = \boxed{-2.749 \times 10^{10}}$

6) Find the sum of the first nine terms of the geometric sequence: $300, 100, \frac{100}{3}, \frac{100}{9}, \ldots$

$r = \frac{1}{3}$

$S_9 = \frac{300(1-(\frac{1}{3})^9)}{(1-(\frac{1}{3}))} \boxed{\approx 449.977}$

7) Find the following sum: $\sum_{i=1}^{10} 6 \cdot 2^i$

8) Find the following sum: $\sum_{i=1}^{8} 2 \cdot 3^i = 6 + 18 + 54 + \cdots$ $r=3$

$S_n = \frac{a_1(1-r^n)}{(1-r)}$

$S_8 = \frac{6(1-3^8)}{(1-3)} = \boxed{19{,}680}$

Computing a Lifetime Salary:

9) A job pays a salary of $30,000 the first year. During the next 29 years, the salary increases by 6% each year. What is the total lifetime salary over the 30-year period?

$100 + 6 \rightarrow 106\%$

$r = 1.06$

$S_n = \frac{a_1(1-r^n)}{(1-r)}$

$S_{30} = \frac{30000(1-1.06^{30})}{(1-1.06)} = \boxed{\$2{,}371{,}749.59}$

What is an annuity? Saving plan → contribute a set amount on a regular basis
→ fixed interest rate
→ "compounded" → calculated mult times per year.

Annuity Formula: $A = \frac{P\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\left(\frac{r}{n}\right)}$

amount of \$ at the end

principle (contributing amount each time)

r = interest rate (as a decimal)
t = # of years
n = # of times compounded

Example 10) At age 30, to save for retirement, you decide to deposit \$100 at the end of each month into an IRA that pays 9.5% compounded monthly. $r = 0.095$ $t = 35$ $n = 12$

Compounded
- monthly $n = 12$
- annually $n = 1$
- weekly $n = 52$
- daily $n = 360$

a) How much will you have from the IRA when you retire at age 65?

$$A = \frac{100\left[\left(1+\frac{0.095}{12}\right)^{12\cdot 35}-1\right]}{\left(\frac{0.095}{12}\right)} \approx \boxed{\$\,333{,}946.30}$$

b) Find the amount of interest earned.

$$\underbrace{100\cdot 12\cdot 35}_{\text{amount contributed}} = \$42{,}000$$

$$333946.30 - 42{,}000 = \$\,291{,}946.30$$

Sum of an Infinite Geometric Series: If $-1 < r < 1$, then the sum of the infinite geometric series is given by $|r|<1$

$$\boldsymbol{S = \frac{a_1}{1-r}}$$

infinite sum

Exploration: What if $|r| \geq 1$? Would the infinite series have a sum? Explain.

3, 6, 12, 24, 48, 96, 192, ...
3 + 6 + 12 + 24 + 48 + 96 + 192 + ...

No, grows without bound

Example 11) Find the sum of the infinite geometric series: $\frac{3}{8} - \frac{3}{16} + \frac{3}{32} - \frac{3}{64} + \cdots$

$$S = \frac{a_1}{1-r} = \frac{\frac{3}{8}}{\frac{2}{2}+\left(+\frac{1}{2}\right)} = \frac{\frac{3}{8}}{\frac{3}{2}} = \frac{3}{8}\cdot\frac{2}{3} = \boxed{\frac{1}{4}}$$

$r = -\frac{1}{2}$ $|-|<1$

Example 12) Find the sum of the infinite geometric series: $3 + 2 + \frac{4}{3} + \frac{8}{9} \ldots$

$r = \frac{a_2}{a_1} = \frac{2}{3}$

$$S = \frac{a_1}{1-r} = \frac{3}{1-\frac{2}{3}} = \frac{3}{\frac{1}{3}} = 3\cdot\frac{3}{1} = \boxed{9}$$

$r = \frac{2}{3}$ $|r|<1$

Example 13) Express $0.\bar{8}$ as a fraction in lowest terms.

0.888888...

$\frac{8}{10} + \frac{8}{100} + \frac{8}{1000} + \frac{8}{10000}$

$r = \frac{1}{10}$ $|r|<1$

$$S = \frac{\frac{8}{10}}{\frac{10}{10}-\frac{1}{10}} = \frac{\frac{8}{10}}{\frac{9}{10}} = \boxed{\frac{8}{9}}$$

10.5 Notes: The Binomial Theorem

Combination: ${}_nC_r$ or $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ Note: for non-negative integers n and r, with $n \geq r$.

Calc
$\boxed{n}$
math, Prob, 3
$\boxed{r}$

total # items selected

${}_{10}C_4$ or $\binom{10}{4} = \frac{10!}{4!(6)!} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7 \cdot \cancel{6!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{6!}} = 210$

Examples 1 – 4: Evaluate each combination:

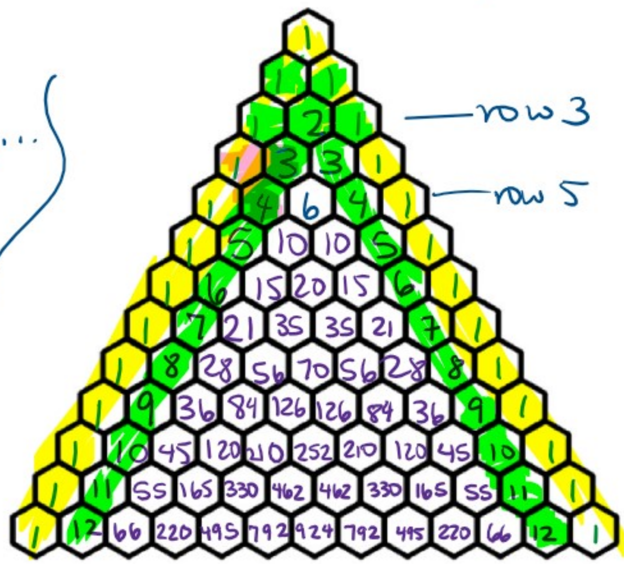
1. $\binom{6}{4} = 15$ 2. $\binom{6}{0} = 1$ 3. $\binom{8}{2} = 28$ 4. $\binom{3}{3}$

On the calculator: $\binom{n}{r} =$ $\boxed{n}$ math, Prob, 3 $\boxed{r}$

Pascal's Δ

Pascal's Triangle:

1, 1, 2, 3, 5, 8, 13, 21, ...
Fibonacci Sequence



Binomial Expansion:

2 terms $(a+b)^n \rightarrow$ mult & combine like terms

Examples 1 – 3: Expand each binomial.

1) $(x+2)^2$

$(x+2)(x+2)$

$x^2 + 4x + 4$

x^2 $2 \cdot 2 \cdot x$ 2^2

2) $(x-5)^2$

$(x-5)(x-5)$

$x^2 - 10x + 25$

$2 \cdot -5 \cdot x$

3) $(4m+3n)^2$

$(4m)^2$ $2 \cdot 4m \cdot 3n$ $(3n)^2$

$= 16m^2 + 24mn + 9n^2$

Exploration: Fill in the following table. ***Hint:** to expand $(x+y)^3$, you can multiply $(x+y)^2$ by $(x+y)^1$.

	Binomial	Expansion
1	$(x+y)^0$	1
2	$(x+y)^1$	$x^1 + y^1$
3	$(x+y)^2$	$x^2 + 2xy + y^2$
4	$(x+y)^3$	$x^3 + 3x^2y + 3xy^2 + y^3$
5	$(x+y)^4$	$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
6	$(x+y)^5$	$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

- Write conjectures about the number of terms and about symmetry in the terms of the expansion in any row of the table. Verify your conjectures by filling in the row that would follow.

The row # is one more than the degree (highest power)
The # of terms is the same as the row #
Each row has symmetry

- Compare your expansions and Pascal's Triangle. Write down your observations below.

SAME as the coeff!

- Use the pattern you saw to expand $(x+y)^{10}$. look at row 11

$1x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6$
$+ 120x^3y^7 + 45x^2y^8 + 10xy^9 + 1y^{10}$

Examples 4 – 6: Expand the following binomials.

4) $(x+2)^4$ use row 5

$1x^4 + 4x^3(2)^1 + 6x^2(2)^2 + 4x(2)^3 + 1(2)^4$

$\boxed{x^4 + 8x^3 + 24x^2 + 32x + 16}$

5) $(x-2y)^5$ use row 6

$1x^5 + 5x^4(-2y) + 10x^3(-2y)^2 + 10x^2(-2y)^3 + 5x(-2y)^4 + 1(-2y)^5$

$\boxed{x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5}$

6) $(3x+2y)^5$ use row 6

$1(3x)^5 + 5(3x)^4(2y) + 10(3x)^3(2y)^2 + 10(3x)^2(2y)^3 + 5(3x)(2y)^4 + 1(2y)^5$

$\boxed{243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240xy^4 + 32y^5}$

Binomial Coefficient: Combinations can be used to find the coefficients for each term when a binomial is expanded. These same values can also be found in Pascal's Triangle.

Finding a particular term in a Binomial Expansion:

The $(r+1)^{st}$ term of the expansion of $(a+b)^n = \binom{n}{r}a^{n-r}b^r$

Examples 7 – 8: Find the indicated term in each expansion.

7) 5^{th} term of $(2x+y)^9$

$5 = r+1$
$4 = r$
$9 = n$

$\binom{9}{4}(2x)^5(y)^4$

$126(32x^5)(y^4)$

$\boxed{4032x^5y^4}$

8) 7^{th} term of $(x-2)^{10}$

$7 = r+1$
$6 = r$
$10 = n$

$\binom{10}{6}(x)^4(-2)^6$

$210x^4 \cdot 64$

$\boxed{13440x^4}$